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ON A MULTISTEP METHOD II

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Abstract :

In the paper [2], we introduced a multistep method for solving the initial value problem :

$$y' = f(x, y),\ y(x_0) = y_0,\ stepsize = h, \tag{0.1}$$

using the idea of quadrature formulas as in [1], for the corrector we use the later part of the formulas, which Prof. W. D. Milne used for the corrector of the starting values.

But, we did not show the algorithm to solve (0.1). So, we want to show that in detail. Next, we renew the predictor for the sake of the accuracy.

1. The new predictor of seven-point formulas

By Taylor's expansion, we have

$$y_{n+j} = y_n + jh\,y_n' + \frac{(jh)^2}{2}y_n'' + \frac{(jh)^3}{6}y_n^{(3)} + \frac{(jh)^4}{24}y_n^{(4)} + \frac{(jh)^5}{120}y_n^{(5)} + \frac{(jh)^6}{720}y_n^{(6)}$$
$$+\frac{(jh)^7}{5040}y_n^{(7)} + \frac{(jh)^8}{40320}y_n^{(8)} + \cdots (j = -3, -2, -1, 1, 2, 3), \tag{1.1}$$

$$y_{n+i}' = y_n' + ihy_n'' + \frac{(ih)^2}{2}y_n^{(3)} + \frac{(ih)^3}{6}y_n^{(4)} + \frac{(ih)^4}{24}y_n^{(5)} + \frac{(ih)^5}{120}y_n^{(6)} + \frac{(ih)^6}{720}y_n^{(7)}$$
$$+\frac{(ih)^7}{5040}y_n^{(8)} + \cdots (i = -3, -2, -1, 1, 2, 3). \tag{1.2}$$

Setting j = 3 in (1.1), and adding $(1.2) \times a_i h$ (i = −6, −5, −4, −3, −2, −1), we obtain the following equation :

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$$y_{n+3}+\sum_{i=-6}^{-1} a_i y'_{n+i}h=y_n+h\left(3+\sum_{i=-6}^{-1} a_i\right)y'_n+h^2\left(\frac{3^2}{2}+\sum_{i=-6}^{-1} ia_i\right)y''_n$$

$$+\frac{h^3}{2}\left(\frac{3^3}{3}+\sum_{i=-6}^{-1} i^2a_i\right)y_n^{(3)}+\frac{h^4}{6}\left(\frac{3^4}{4}+\sum_{i=-6}^{-1} i^3a_i\right)y_n^{(4)}$$

$$+\frac{h^5}{24}\left(\frac{3^5}{5}+\sum_{i=-6}^{-1} i^4a_i\right)y_n^{(5)}+\frac{h^6}{120}\left(\frac{3^6}{6}+\sum_{i=-6}^{-1} i^5a_i\right)y_n^{(6)}$$

$$+\frac{h^7}{720}\left(\frac{3^7}{7}+\sum_{i=-6}^{-1} i^6a_i\right)y_n^{(7)}+\frac{h^8}{5040}\left(\frac{3^8}{8}+\sum_{i=-6}^{-1} i^7a_i\right)y_n^{(8)}+\cdots. \qquad (1.3)$$

Setting zero the coefficients of $y_n^{(k)}$ $(k=2, 7)$ at the equation above, we obtain the following equation:

$$AX=b, \qquad (1.4)$$

where

$$A=\begin{pmatrix} -6 & -5 & -4 & -3 & -2 & -1 \\ 36 & 25 & 16 & 9 & 4 & 1 \\ -216 & -125 & -64 & -27 & -8 & -1 \\ 1296 & 625 & 256 & 81 & 16 & 1 \\ -7776 & -3125 & -1024 & -243 & -32 & -1 \\ 46656 & 15625 & 4096 & 729 & 64 & 1 \end{pmatrix},$$

X is the unknown six dimentional vector, and

$$b=\left(-\frac{9}{2}, -\frac{27}{3}, -\frac{81}{4}, -\frac{243}{5}, -\frac{729}{6}, -\frac{2187}{7}\right)^t$$

Solving this equation (1.4), we obtain the following equation:

$$y_{n+3}=y_n+\frac{h}{2240}\left(43021y'_{n-6}-293112y'_{n-5}+847881y'_{n-4}-1341824y'_{n-3}+1239111y'_{n-2}\right.$$
$$\left.-646920y'_{n-1}+158563y'_n\right)+\frac{103437}{4480}h^8y_n^{(8)}. \qquad (1.5)$$

For the sake of the comparison with our old formulas, we rewrite our corrector formulas and predictor formula at [2].

The correctors:

$$y_{n-3}=y_n-\frac{h}{2240}(685y'_{n-3}+3240y'_{n-2}+1161y'_{n-1}+2176y'_n$$

$$-729y'_{n+1}+216y'_{n+2}-29y'_{n+3})-\frac{9}{896}h^8y_n^{(8)}, \tag{1.6.1}$$

$$y_{n-2}=y_n-\frac{h}{3780}(-37y'_{n-3}+1398y'_{n-2}+4863y'_{n-1}+1328y'_n$$
$$+33y'_{n+1}-30y'_{n+2}+5y'_{n+3})+\frac{1}{756}h^8y_n^{(8)}, \tag{1.6.2}$$

$$y_{n-1}=y_n-\frac{h}{60480}(271y'_{n-3}-2760y'_{n-2}+30819y'_{N-1}+37504y'_n$$
$$-6771y'_{n+1}+1608y'_{n+2}-191y_{n+3})+\frac{191}{120960}h^8y_n^{(8)}, \tag{1.6.3}$$

$$y_{n+1}=y_n+\frac{h}{60480}(-191y'_{n-3}+1608y'_{n-2}-6771y'_{n-1}+37504y'_n$$
$$+30819y'_{n+1}-2760y'_{n+2}+271y'_{n+3})-\frac{191}{120960}h^8y_n^{(8)}, \tag{1.6.4}$$

$$y_{n+2}=y_n+\frac{h}{3780}(5y'_{n-3}-30y'_{n-2}+33y'_{n-1}+1328y'_n$$
$$+4863y'_{n+1}+1398y'_{n+2}-37y'_{n+3})+\frac{h^8}{756}y_n^{(8)}, \tag{1.6.5}$$

$$y_{n+3}=y_n+\frac{h}{2240}(-29y'_{n-3}+216y'_{n-2}-729y'_{n-1}+2176y'_n$$
$$+1161y'_{n+1}+3240y'_{n+2}+685y'_{n+3})-\frac{9}{896}h^8y_n^{(8)}. \tag{1.6.6}$$

The predictor:

$$y_{n+4}=y_n+\frac{h}{945}(286y'_{n-3}-2010y'_{n-2}+6054y'_{n-1}-9836y'_n+11514y'_{n+1}$$
$$-5622y'_{n+2}+3394y'_{n+3})+\frac{278}{945}h^8y_n^{(8)}. \tag{1.7}$$

2. The new predictor of six-point formulas

Using Taylor's expansion, we have

$$y_{n+j}=y_n+jhy'_n+\frac{(jh)^2}{2}y''_n+\frac{(jh)^3}{6}y_n^{(3)}+\frac{(jh)^4}{24}y_n^{(4)}+\frac{(jh)^5}{120}y_n^{(5)}$$
$$+\frac{(jh)^6}{720}y_n^{(6)}+\frac{(jh)^7}{5040}y_n^{(7)}+\cdots \quad (j=-3,-2,-1,1,2), \tag{2.1}$$

$$y'_{n+i}=y'_n+ihy''_n+\frac{(ih)^2}{2}y_n^{(3)}+\frac{(ih)^3}{6}y_n^{(4)}+\frac{(ih)^4}{24}y_n^{(5)}$$

$$+\frac{(ih)^5}{120}y_n^{(6)}+\frac{(ih)^6}{720}y_n^{(7)}+\cdots(i=-3,-2,-1,1,2). \tag{2.2}$$

Setting $j=2$ at (2.1), and adding $(2.2)\times h\times a_i$ $(i=-5,-4,-3,-2,-1)$ we have :

$$y_{n+2}+\sum_{i=-5}^{-1}a_iy'_{n+i}h=y_n+h\left(2+\sum_{i=-5}^{-1}a_i\right)y'_n+h^2\left(\frac{2^2}{2}+\sum_{i=-6}^{-1}ia_i\right)y''_n$$

$$+\frac{h^3}{2}\left(\frac{2^3}{3}+\sum_{i=-5}^{-1}i^2a_i\right)y_n^{(3)}+\frac{h^4}{6}\left(\frac{2^4}{4}+\sum_{i=-5}^{2}i^3a_i\right)y_n^{(4)}$$

$$+\frac{h^5}{24}\left(\frac{2^5}{5}+\sum_{i=-5}^{-1}i^4a_i\right)y_n^{(5)}+\frac{h^6}{120}\left(\frac{2^6}{6}+\sum_{i=-5}^{-1}i^5a_i\right)y_n^{(6)}$$

$$+\frac{h^7}{720}\left(\frac{2^7}{7}+\sum_{i=-5}^{-1}i^6a_i\right)y_n^{(7)}+\cdots. \tag{2.3}$$

Setting zero the coefficients of $y_n^{(k)}$ $(k=2,6)$, we have the following equation :

$$AX=b, \tag{2.4}$$

where

$$A=\begin{pmatrix} -5 & -4 & -3 & -2 & -1\\ 25 & 16 & 9 & 4 & 1\\ -125 & -64 & -27 & -8 & -1\\ 625 & 256 & 81 & 16 & 1\\ -3125 & -1024 & -243 & -32 & -1\end{pmatrix},$$

X is the unknown five dimentional vector, and

$$b=\left(-\frac{4}{2},-\frac{8}{3},-\frac{16}{4},-\frac{32}{5},-\frac{64}{6}\right)^t$$

Solving this equation (2.4), we obtain the following equation :

$$y_{n+2}=y_2+\frac{h}{90}(-297y'_{n-5}+1754y'_{n-4}-4286y'_{n-3}+5514y'_{n-2}+3881y'_{n-1}+1376y'_n)+\frac{13613}{3780}h^7y_n^{(7)} \tag{2.5}$$

For the sake of the comparison with our old formulas, we rewrite our corrector formulas and predictor formula at [2].

The correctors:

$$y_{n-3}=y_n-\frac{h}{160}(51y'_{n-3}+219y'_{n-2}+114y'_{n-1}+114y'_n-21y'_{n+1}+3y'_{n+2})$$

$$+\frac{29}{2240}h^7y_n^{(7)}, \qquad (2.6.1)$$

$$y_{n-2}=y_n-\frac{h}{90}(-y'_{n-3}+34y'_{n-2}+114y'_{n-1}+34y'_n-y'_{n+1})-\frac{1}{756}h^7y_n^{(7)}, \qquad (2.5.2)$$

$$y_{n-1}=y_n-\frac{h}{1440}(11y'_{n-3}-93y'_{n-2}+802y'_{n-1}+802y'_n-93y'_{n+1}+11y'_{n+2})$$

$$+\frac{191}{60480}h^7y_n^{(7)}, \qquad (2.6.3)$$

$$y_{n+1}=y_n+\frac{h}{1440}(-11y'_{n-3}+77y'_{n-2}-258y'_{n-1}+1022y'_n+637y'_{t+1}-27y'_{n+2})$$

$$+\frac{271}{60480}h^7y_n^{(7)}, \qquad (2.6.4)$$

$$y_{n+2}=y_n+\frac{h}{90}(y'_{n-3}-6y'_{n-2}+14y'_{n-1}+14y'_n+129y'_{n+1}+28y'_{n+2})$$

$$-\frac{37}{3780}h^7y_n^{(7)}, \qquad (2.6.5)$$

The predictor:

$$y_{n+3}=y_n+\frac{h}{160}(-51y'_{n-3}+309y'_{n-2}-786y'_{n-1}+1134y'_n-651y'_{n+1}+525y'_{n+2})$$

$$+\frac{137}{448}h^7y_n^{(7)}. \qquad (2.7)$$

3. The new predictor of five-point formulas

Using Taylor's expansion, we have

$$y_{n+j}=y_n+jhy'_n+\frac{(jh)^2}{2}y''_n+\frac{(jh)^3}{6}y_n^{(3)}+\frac{(jh)^4}{24}y_n^{(4)}+\frac{(jh)^5}{120}y_n^{(5)}$$

$$+\frac{(jh)^6}{720}y_n^{(6)}+\cdots(j=-2,-1,1,2), \qquad (3.1)$$

$$y'_{n+i}=y'_n+ihy''_n+\frac{(ih)^2}{2}y_n^{(3)}+\frac{(ih)^3}{6}y_n^{(4)}+\frac{(ih)^4}{24}y_n^{(5)}+\frac{(ih)^5}{120}y_n^{(6)}$$

$$+\cdots\ (i=-2,\ -1,\ 1,\ 2). \tag{3.2}$$

Setting $j=2$ at (3.1), and adding (3.2)$\times h\times a_i$ $(i=-4,\ -3,\ -2,\ -1)$, we have:

$$y_{n+2}+\sum_{i=-4}^{-1} a_i y'_{n+i}h=y_n+h\left(2+\sum_{i=-4}^{-1} a_i\right)y'_n+h^2\left(\frac{2^2}{2}+\sum_{i=-4}^{-1} ia_i\right)y''_n$$

$$+\frac{h^3}{2}\left(\frac{2^3}{3}+\sum_{i=-4}^{-1} i^2a_i\right)y_n^{(3)}+\frac{h^4}{6}\left(\frac{2^4}{4}+\sum_{i=-4}^{-1} i^3a_i\right)y_n^{(4)}$$

$$+\frac{h^5}{24}\left(\frac{2^5}{5}+\sum_{i=-4}^{-1} i^4a_i\right)y_n^{(5)}+\frac{h^6}{120}\left(\frac{2^6}{6}+\sum_{i=-4}^{-1} i^5a_i\right)y_n^{(6)}+\cdots. \tag{3.3}$$

Setting zero the coefficients of $y_n^{(k)}$ $(k=2,\ 5)$, we have the equation.

$$AX=b, \tag{3.4}$$

where

$$A=\begin{pmatrix} -4 & -3 & -2 & -1 \\ 16 & 9 & 4 & 1 \\ -64 & -27 & -8 & -1 \\ 256 & 81 & 16 & 1 \end{pmatrix},$$

X is the unknown four dimentional vector, and

$$b=\left(-\frac{4}{2},\ -\frac{8}{3},\ -\frac{16}{4},\ -\frac{32}{5}\right)^t.$$

Solving this equation (3.4), we obtain the following equation:

$$y_{n+2}=y_n+\frac{h}{90}\left(269y'_{n-4}-1316y'_{n-3}+2544y'_{n-2}-2369y'_{n-1}+1079y'_n\right)+\frac{33}{10}h^6y_n^{(6)}. \tag{3.5}$$

For the sake of the comparison with our old formulas, we rewrite the corrector formulas and the predictor formula at [2].

The correctors:

$$y_{n-2}=y_n-\frac{h}{90}\left(29y'_{n-2}+124y'_{n-1}+24y'_n+4y'_{n+1}-y'_{n+2}\right)-\frac{1}{90}h^6y_n^{(6)}, \tag{3.6.1}$$

$$y_{n-1}=y_n-\frac{h}{720}\left(-19y'_{n-2}+346y'_{n-1}+456y'_n-74y'_{n+1}+11y'_{n+2}\right)+\frac{11}{1440}h^6y_n^{(6)}, \tag{3.6.2}$$

$$y_{n+1}=y_n+\frac{h}{720}(11y'_{n-2}-74y'_{n-1}+456y'_n+346y'_{n+1}-19y'_{n+2})-\frac{11}{1440}h^6y_0^{(6)}, \qquad (3.6.3)$$

$$y_{n+2}=y_n+\frac{h}{90}(-y'_{n-2}+4y'_{n-1}+24y'_n+124y'_{n+1}+29y'_{n+2})-\frac{1}{90}h^6y_n^{(6)}, \qquad (3.6.4)$$

The predictor:

$$y_{n+3}=y_n+\frac{h}{80}(27y'_{n-2}-138y'_{n-1}+312y'_n-198y'_{n+1}+237y'_{n+2})+\frac{51}{160}h^6y_n^{(6)}. \qquad (3.7)$$

4. The algorithms by our formulas

We explain the algorithm of our formulas for the each cases

(a) the seven-point formulas

(1) Setting **eps** $=10^{-k}$, where k depends to the step-size h, we use this value for the check of the convergence.

(2) By the Runge-Kutta Method, we get y^0_{-3}, y^0_{-2}, y^0_{-1}, y^0_1, y^0_2, and y^0_3. Then, using the given differential equation, we have y'^0_i $(i=-3, 3)$ $(i\neq 0)$.

(3) By (1.5.1) to (1.5.6), we correct y^0_i $(i=-3, 3)$ $(i\neq 0)$ to y_i $(i=-3, 3)$ $(i\neq 0)$, till these values converge.

(4) When y_i and y'_i $(i=n, n-1, n-2, \ldots)$ are decided, we write y_n by Y0D, y'_n by DY0D, y_{n-k} by YMkD $(k=1, 2, \ldots)$, y'_{n-k} by DYMkD $(k=1, 2, \ldots)$, y_{n+1} by Y1C, y'_{n+1} by Dy1C, y_{n+2} by Y2CP, and y'_{n+2} by DY2CP.

(5) By the predictor (1.5), we have y_{n+3}, which is written by Y3P. Next, from the given differential equation, we get y'_{n+3}, which we show by DY3P.

(6) By (1.6.4), (1.6.5), and (1.6.6), we correct y_{n+k} $(k=1, 3)$. Then, we have y'_{n+k} $(k=1, 3)$ by the given differential equation. We write y_{n+k} and y'_{n+k} $(k=1, 3)$ by YkC1 and DYkC1 $(k=1, 3)$ respectively.

(6.1) If $|Y1C1-Y1C|<eps$, $|Y2C1-Y2CP|<eps$, and $|Y3C1-Y3P|<eps$, then we make twise the step size h, if we forward ten steps after the change of stepsize. (Go to (8).)

If we do not forward ten steps, then we decide y_{n+1}. Then, we renumber y_k and y'_k $(k=n+1, n, n-1, \ldots)$ to y_k and y'_k $(k=n, n-1, n-2, \ldots)$. Next, repeat (4) to (6).

(6.2) Otherwise, by (1.6.4), (1.6.5), and (1.6.6), we correct y_{n+k} $(k=1, 3)$. Then, we have y'_{n+k} $(k=1, 3)$ by the given differential equation. We write y_{n+k} and

y'_{n+k} (k=1, 3) by $YkC2$ and $DYkC2$ ($k=1$, 3) respectively.

(6.2.1) If $|Y1C1-Y1C2|<eps$, $|Y2C1-Y2C2|<eps$, and $|Y3C1-Y3C2|<eps$, then we decide y_{n+1}. Next, we renumber y_k and y'_k ($k=n+1, n, n-1, \ldots$) to y_k and y'_k ($k=n, n-1, n-2, \ldots$). Then, repeat (4) to (6).

(6.2.2) Otherwise, by (1.6.4), (1.6.5), and (1.6.6), we correct y_{n+k} ($k=1$, 3). Then, we have y'_{n+k} ($k=1$, 3) by the given differential equation. We write y_{n+k} and y'_{n+k} ($k=1$, 3) by $YkC3$ and $DYkC3$ ($k=1$, 3) respectively.

(6.2.3) If $|Y1C2-Y1C3|<eps$, $|Y2C2-Y2C3|<eps$, and $|Y3C2-Y3C3|<eps$, then we decide y_{n+1}. And, we renumber y_k and y'_k ($k=n+1, n, n-1, \ldots$) to y_k and y'_k ($k=n, n-1, n-2, \ldots$). Then, repeat (4) to (6).

(6.2.4) Otherwise, we separate the stepsize h half, if we forward ten steps after the change of the step size. (Go to (7).) If we do not forward ten steps after the change of the stepsize, we repeat till y_{n+1} congerge. Then, we decide y_{n+1}. Next, we renumber y_k and y'_k ($k=n+1, n, n-1, \ldots$) to y_k and y'_k ($k=n, n-1, n-2, \ldots$). Then, repeat (4) to (6).

(7) In (6.2.4), if we need, we make half the stepsize h, by the following formula :

$$y_{\frac{1}{2}}=\frac{75}{128}(y_0+y_1)-\frac{125}{1280}(y_{-1}+y_2)+\frac{75}{6400}(y_{-2}+y_3). \tag{4.1}$$

We renumber y_n, $y_{n-\frac{1}{2}}$, y_{n-1}, $y_{n-1-\frac{1}{2}}$, y_{n-2}, and $y_{n-2-\frac{1}{2}}$ to y_k($k=2, 1, 0, -1, -2, -3$). Then, we correct y_{n-3}, y_{n-1}, and y_{n+1} by (1.6.1), (1.6.3), and (1.6.4) respectively, till these values converge. Next, repeat (4) to (6).

(8) In (6.1), if we need, we make twice the stepsize. We renumber y_k ($k=n-7, n-5, n-3, n-1, n+1, n+3$) to y_k ($k=$ $(-3, 2)$). Then, we repeat (4) to (6).

(b) the six-point formulas

(1) Setting **eps** $=10^{-k}$, where k depends to the step-size h,we use this value for the check of the convergence.

(2) By the Runge-Kutta Method, we get y^0_{-3}, y^0_{-2}, y^0_{-1}, y^0_1, and y^0_2. Then, using the given differential equation, we have y'^0_i ($i=-3$, 2) ($i\neq 0$).

(3) By (2.5.1) to (2.5.5), we correct y^0_i($i=-3$, 2) ($i\neq 0$) to y_i($i=-3$, 2) ($i\neq 0$), till these values converge.

(4) When y_i and y'_i ($i=n, n-1, n-2, \ldots$) are decided, we write y_n by

Y0D, y'_n by DY0D, y_{n-k} by YMkD ($k=1, 2, \dots$), y'_{n-k} by DYMkD ($k=1, 2, \dots$), y_{n+1} by Y1C, and y'_{n+1} by DY1C.

(5) By the predictor (2.5), we have y_{n+2}, which is written by Y2P. Next, from the given differential equation, we get y'_{n+2}, which we show by DY2P.

(6) By (2.6.4), and (2.6.5), we correct y_{n+k} ($k=1, 2$). Then, we have y'_{n+k} ($k=1, 2$) by the given differential equation. We write y_{n+k} and y'_{n+k} ($k=1, 2$) *by YkC*1 *and DYkC*1 ($k=1, 2$) respectively.

(6.1) If $|Y1C1-Y1C|<eps$ and $|Y2C1-Y2CP|<eps$, then we make twise the step size h. If we forward ten steps after the change of stepsize, we go to (8).

If we do not forward ten steps, then we decide y_{n+1}. Then, we renumber y_k and y'_k ($k=n+1, n, n-1, \dots$) to y_k and y'_k ($k=n, n-1, n-2, \dots$). Next, repeat (4) to (6).

(6.2) Otherwise, by (2.6.4) and (2.6.5), we correct y_{n+k} ($k=1, 2$). Then, we have y'_{n+k} ($k=1, 2$) by the given differential equation. We write y_{n+k} and y'_{n+k} (k=1, 2) by YkC2 and DYkC2 ($k=1, 2$) respectively.

(6.2.1) If $|Y1C1-Y1C2|<eps$ and $|Y2C1-Y2C2|<eps$, then we decide y_{n+1}. Next, we renumber y_k and y'_k ($k=n+1, n, n-1, \dots$) to y_k and y'_k ($k=n, n-1, n-2, \dots$). Then, repeat (4) to (6).

(6.2.2) Otherwise, by (2.6.4) and (2.6.5), we correct y_{n+k} ($k=1, 2$). Then, we have y'_{n+k} ($k=1, 2$) by the given differential equation. We write y_{n+k} and y'_{n+k} ($k=1, 2$) by *YkC*3 and *DYkC*3 ($k=1, 2$) respectively.

(6.2.3) If $|Y1C2-Y1C3|<eps$ and $|Y2C2-Y2C3|<eps$, then we decide y_{n+1} Next, we renumber y_k and y'_k ($k=n+1, n, n-1, \dots$) to y_k and y'_k ($k=n, n-1, n-2, \dots$). Then, repeat (4) to (6).

(6.2.4) Otherwise, we separate the stepsize h half, if we forward ten steps after the change of the step size. We go to (7). If we do not forward ten steps after the change of the stepsize, we repeat till y_{n+1} converge. Then, we decide y_{n+1}. Next, we renumber y_k and y'_k ($k=n+1, n, n-1, \dots$) to y_k and y'_k ($k=n, n-1, n-2, \dots$). Then, repeat (4) to (6).

(7) In (6.2.4), if we need, we make half the stepsize h. We get $y_{n+\frac{1}{2}}$, $y_{n-\frac{1}{2}}$, and $y_{n-1-\frac{1}{2}}$, by the following formula:

$$y_{\frac{1}{2}}=\frac{75}{128}(y_0+y_1)-\frac{125}{1280}(y_{-1}+y_2)+\frac{75}{6400}(y_{-2}+y_3). \tag{4.1}$$

We renumber $y_{n+\frac{1}{2}}$, y_n, $y_{n-\frac{1}{2}}$, y_{n-1}, and $y_{n-1-\frac{1}{2}}$ to y_k($k=1, 0, -1, -2, -3$). Then, we correct y_{n-3}, y_{n-1}, and y_{n+1} by (2.6.1), (2.6.3), and (2.5.4) respectively, till these values converge. Next, repeat (4) to (6).

(8) In (6.1), if we need, we make twice the stepsize. We renumber y_k $(k=n-6,\ n-4,\ n-2,\ n,\ n+2)$ to y_k $(k=\ (-3,\ 1))$. Then, we repeat (4) to (6).

(c) the five-point formulas

(1) Setting **eps** $=10^{-k}$, where k depends to the step-size h, we use this value for the check of the convergence.

(2) By the Runge-Kutta Method, we get y^0_{-2}, y^0_{-1}, y^0_1, and y^0_2. Then, using the given differential equation, we have y'^0_i $(i=-2,\ 2)$ $(i\neq 0)$.

(3) By (3.5.1) to (3.5.4), we correct y^0_i $(i=-2,\ 2)$ $(i\neq 0)$ to y_i $(i=-2,\ 2)$ $(i\neq 0)$, till these values converge.

(4) When y_i and y'_i $(i=n,\ n-1,\ n-2,\ldots)$ are decided, we write y_n by Y0D, y'_n by DY0D, y_{n-k} by YMkD $(k=1,\ 2,\ldots)$, y'_{n-k} by DYMkD $(k=1,\ 2,\ldots)$, y_{n+1} by Y1C, and y'_{n+1} by DY1C.

(5) By the predictor (3.5), we have y_{n+2}, which is written by Y2P. Next, from the given differential equation, we get y'_{n+2}, which we show by DY2P.

(6) By (3.6.3), and (3.6.4), we correct y_{n+k} $(k=1,\ 2)$. Then, we have y'_{n+k} $(k=1,\ 2)$ by the given differential equation. We write y_{n+k} and y'_{n+k} $(k=1,\ 2)$ by YkC1 and DYkC1 $(k=1,\ 2)$ respectively.

(6.1) If $|Y1C1-Y1C|<eps$ and $|Y2C1-Y2CP|<eps$, then we make twise the step size h. If we forward ten steps after the change of stepsize, we go to (8).

If we do not forward ten steps, then we decide y_{n+1}. Next, we renumber y_k and y'_k $(k=n+1,\ n,\ n-1,\ldots)$ to y_k and y'_k $(k=n,\ n-1,\ n-2,\ldots)$. Then, repeat (4) to (6).

(6.2) Otherwise, by (3.6.3) and (3.6.4), we correct y_{n+k} $(k=1,\ 2)$. Then, we have y'_{n+k} $(k=1,\ 2)$ by the given differential equation. We write y_{n+k} and y'_{n+k} $(k=1,\ 2)$ by YkC2 and DYkC2 $(k=1,\ 2)$ respectively.

(6.2.1) If $|Y1C1-Y1C2|<eps$ and $|Y2C1-Y2C2|<eps$, then we decide y_{n+1} Next, we renumber y_k and y'_k $(k=n+1,\ n,\ n-1,\ldots)$ to y_k and y'_k $(k=n,\ n-1,\ n-2,\ldots)$. Then, repeat (4) to (6).

(6.2.2) Otherwise, by (3.6.3) and (3.6.4), we correct y_{n+k} $(k=1,\ 2)$. Then, we have y'_{n+k} $(k=1,\ 2)$ by the given differential equation. We write y_{n+k} and y'_{n+k} $(k=1,\ 2)$ by $YkC3$ and $DYkC3$ $(k=1,\ 2)$ respectively.

(6.2.3) If $|Y1C2-Y1C3|<eps$ and $|Y2C2-Y2C3|<eps$, then we decide y_{n+1}. Next, we renumber y_k and y'_k ($k=n+1$, n, $n-1, \ldots$) to y_k and y'_k ($k=n$, $n-1$, $n-2, \ldots$). Then, repeat (4) to (6).

(6.2.4) Otherwise, we separate the stepsize h half, if we forward ten steps after the change of the step size. We go to (7). If we do not forward ten steps after the change of the stepsize, we repeat till y_{n+1} converge. Then, we decide y_{n+1}. Next, we renumber y_k and y'_k ($k=n+1$, n, $n-1, \ldots$) to y_k and y'_k ($k=n$, $n-1$, $n-2, \ldots$). Then, repeat (4) to (6).

(7) In (6.2.4), if we need,we make half the stepsize h. We get $y_{n+\frac{1}{2}}$ and $y_{n-\frac{1}{2}}$, by the following formula :

$$y_{\frac{1}{2}}=\frac{75}{128}(y_0+y_1)-\frac{125}{1280}(y_{-1}+y_2)+\frac{75}{6400}(y_{-2}+y_3). \qquad (4.1)$$

We renumber $y_{n+\frac{1}{2}}$, y_n, $y_{n-\frac{1}{2}}$, and y_{n-1} to y_k ($k=1, 0, -1, -2$). Then, we correct y_{n-1} and y_{n+1} by (3.6.2) and (3.6.3) respectively, till these values converge. Next, repeat (4) to (6).

(8) In (6.1), if we need, we make twice the stepsize. We renumber y_k ($k=n-4$, $n-2$, n, $n+2$) to y_k ($k=(-2, 1)$). Then, we repeat (4) to (6).

References

[1] W. D. Milne: Numerical Solution of Differential Equations, John Wiley and Sons, Inc., New York, 1953.

[2] Y. Inamasu, Y. Kaneko, and H. Yanagihara: On a Multistep Method, J. of the Faculty of International Studies of Culture, Kyushu Sangyo Univ., No. 1, pp. 179–196, Dec. 1994.