Capital accumulation paths and their bifurcations in neoclassical economic growth model

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《INTRODUCTION》

In recent decades,the analytical methods of Dynamical Macroeconomics are often making use of Nonlinear Dynamics such as the theory of Chaos and the theory of Bifurcation. And analytical framework of many of dynamical macroeconomics is based on the neoclassical growth theory rather than on the dynamic counterpart of Keynesian economics. It is true that neoclassical growth models have been applied mainly to the long term problems such as economic developments. But neoclassical growth models nowadays are being used as a framework of the short term economic problems such as fiscal and monetary policies. This fact might be mainly due to the reason that the analysis of intertemporal optimal behavior of economic agents is necessary for the sound microfoundation of dynamic macroeconomics. Neoclassical growth theories are more suitable for this object than Keynesian economics.

Neoclassical growth models that are being applied to dynamical macroeconomics are roughly speaking consisted of three main analytical streams. These streems are the descriptive growth models based on R.Solow (1956) and the optimal growth models based on F.Ramsey (1928) and the overlapping generations models based on P.Diamond (1965). There already appeared several excellent surveys concerning these new developments in dynamical macroeconomics. For example,Nishimura and Yano (1993) surveyed the analyses which deal with the capital accumulation paths and the phenomena of choas in the framework of the optimal growth model. And C.Azariadis (1993) surveyed comprehensively dynamical macroeconomics in the framework of the overlapping generations model.

In this paper,I study the dynamic evolution of capital accumulation paths and their bifurcations in which the production function is nonconcave. Dechert and Nishimura (1983) studied the properties of the capital accumulation paths in the model of a nonconcave production function in the framework of the optimal growth model. They found that there exists a critical level of capital stock and any capital accumulation path starting from above this critical level converges monotonously to some steady state equilibrium. Galor and Ryder (1989) studied,in the framework of the overlapping generations model,the conditions of the production function which are necessary for the capital accumulation paths to converge to some positive steady state. They showed that so called Inada condition is not sufficent to have this property.

In this paper,in the framework of the descriptive growth model of Solow type and of the overlapping generations model,I will clarify the capital accumulation paths in which the production function has a part of increasing returns,and their bifurcations when the parameters of the model change. And as an application,I consider the vicious cycle of poverty when the production function is nonconcave.

This paper consists of 6 sections. In the first section,I explain the framework of the descriptive growth model of Solow type and derive the basic accumulation function. In the second section,in the framework of the descriptive growth model,I consider the properties of the capital accumulation paths and their bifurcations when the production function is S shape. In the third section,I study an example economy of the second section in which the production function is specified. In the fourth section,in the context of the vicious cycle of poverty,I consider the capital accumulation paths and their bifurcations when the production function is M shape. In the fifth section,in the framework of the overlapping generations model,I consider the same problems of the second section. In the sixth section,I study an example economy of the fifth section in which the production function and the utility function is specified.

《SECTION 1》

The capital accumulation paths of the descriptive growth model of Solow type.

First of all,I set up the capital accumulation model of Solow type in terms of descrete time analysis. In this aggregated one sector model, output Y is produced by using the services of caital stock K and labor L. Output is either consumed or saved to become capital accumulation. The propensity s of savings out of income is assumed to be a constant parameter. The aggregated prodction function F(K,L) is supposed to be neoclassical. That is to say, $F(K,L)$ is continuously differentiable, increasing function in K and L. And $F(K,L)$ is also lineary homogenous, strictly concave function. K and L is thought to be indispensable in producing goods, namely

(1.1) $F(K,0)=0, (\forall\ K\geqq 0);\ F(0,L)=0, (\forall\ L\geqq 0).$

When economic variables are expressed in terms of variables per labor unit,the assumptions mentioned become

(1.2) $f(0)=0,\ f'(k)>0,\ f''(k)<0,$

where k is capital stock per labor unit, $f'(k)$ and $f''(k)$ are respectivly the first and second derivatives in k. It is natural to think that when capital stock is relatively scarce compared to labor, the marginal productivity of capital is sufficently large and vice versa. Therfore we can assume,for example

(1.3) $0\leqq \lim_{k\to\infty} f'(k) < (\delta+n)/s < \lim_{k\to 0} f'(k) \leqq \infty,$

where δ is the rate of deterioration in capital stock and n is the rate of growth of labor.

The current (t period) net investment and the replacement investment are respectively expressed as $K_{t+1}-K_t$ and δK_t. Therefore the gross investment in terms of per labor unit is

(1.4) $(K_{t+1}-K_t+\delta K_t)/L_t=K_{t+1}/L_t-(1-\delta)K_t/L_t=(1+n)k_{t+1}-(1-\delta)k_t.$

The resources for this goss investment is supplied by the savings out of current income. As the propensity of savings out of income is assumed to be a constant,the current savings per labor is $sf(k_t)$ at period t. So we have a basic relation of capital accumulation

(1.5) $(1+n)k_{t+1}-(1-\delta)k_t=sf(k_t)$.

This equation is arranged to

(1.6) $k_{t+1}=\{(1-\delta)k_t+sf(k_t)\}/(1+n)\equiv h(k_t)$.

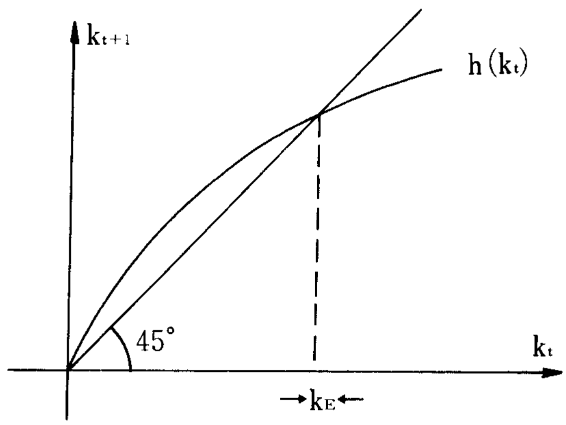
The $h(\cdot)$ function defined in this way is called as the capital accumulation function in the descriptive growth model of Solow type.

Under the assumptions of (1.2) and (1.3),it is easy to see

(1.7) $h(0)=0,\ h'(k)>0,\ h''(k)<0,\ \lim_{k\to 0}h'(k)>1,\ \lim_{k\to\infty}h'(k)<1$.

The basic relation of capital accumulation of (1.6) is represented on the phase diagram 1—1. We can easily see from this diagram that any capital accumulation path starting from any initial point except k_E wll converge globally to the positive steady state k_E.

diagram 1 — 1



Differentiating $h(\cdot)$ function at the steady state k_E,we get

(1.8) $\partial k_E/\partial s=f/(n+\delta-sf')>0,\ \partial k_E/\partial n=\partial k_E/\partial\delta=-k_E/(n+\delta-sf')<0$,

where the relation $(n+\delta-sf')>0$ comes from the stability condition $h'(k_E)<1$ of the capital accumulation process of (1.6).

《SECTION 2》

The capital accumulation paths when the production function has a part of increasing returns;the production function is S shape.

In this section,we assume the number of labor is constant and the production function has a part of increasing returns. The framework of the model in this section is quite similar to that of section 1 except the production function. We express the production function in this section as $g(k)$ and assme

(2.1)　$g(0)=0,\ g'(k)>0,\ \lim_{k\to\infty} g'(k)=0.$

$g''(k)>0\ (0<k<k_I),\ g''(k)=0\ (k=k_I),\ g''(k)<0\ (k>k_I).$

This means that the production function $g(k)$ is S shape and has a reflection point at k_I. This type of the production function is often used as a short term production relation. We assumed that the number of labor is constant,because in the short run the change of capital stock is relatively larger than that of labor and in advanced nations the rate of change in population is quite small.

Like the analysis in the section 1,the current gross investment is $k_{t+1}-k_t+\delta k_t$ and this resource of the innvestment is supplied by the savings of the economy,namely $sg(k_t)$. Consequently we have

(2.2) $k_{t+1}-k_t+\delta k_t=sg(k_t).$

This is arranged to

(2.3) $k_{t+1}=(1-\delta)k_t+sg(k_t)\equiv h(k_t).$

The $h(k_t)$ function thus defined is the capital accumulation function in this section. According to the assumption (2.1),it is easy to see

(2.4) $h(0)=0,\ h'(k)=1-\delta+sg'(k)>0,\ \lim_{k\to\infty}h'(k)=1-\delta<1,$

$h''(k)=sg''(k).$

By making use of an analysis in the phase diagrams,we can clarify the qualitaive properties of the capital accumulation paths. At first, depending on the condition of $h'(0)>1$ or $h'(0)<1$,we can devide two cases of the capital accumulation paths. The case $h'(0)>1$ is shown in ① of the diagram 2-1,and the qaulitative nature is the same as that of section 1. On the other hand,the case $h'(0)<1$ is shown in ② of the diagram 2-1,and there might arise zero or one or two positive steady states. As for this case,arranging the phase diagrams in order of productivity,we have the diagram 2-2.

diagram 2 — 1 ①

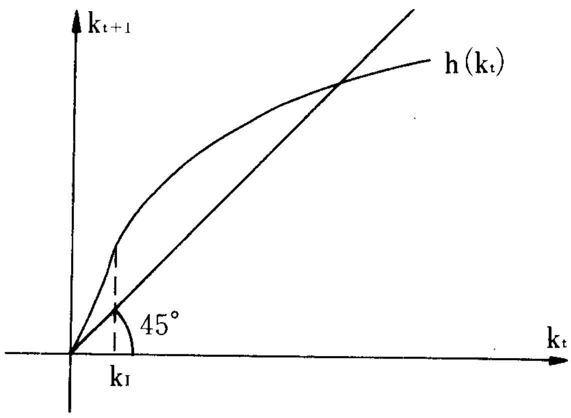
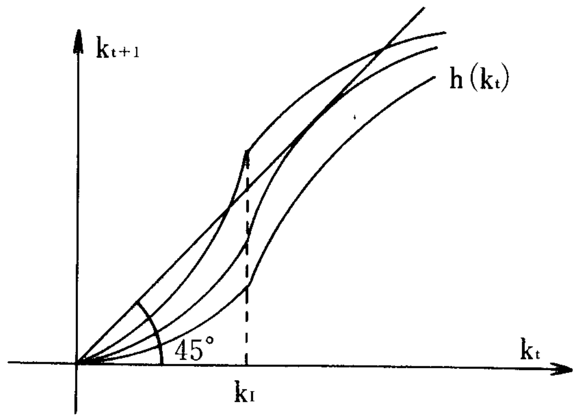


diagram 2 — 1 ②



In the case of ① of this diagram,the productivity of an economy is so low that any capital accumulation path starting from any positive initial stock will converge globally to zero capital stock,namely abusolute poverty.

diagram 2 — 2 ①

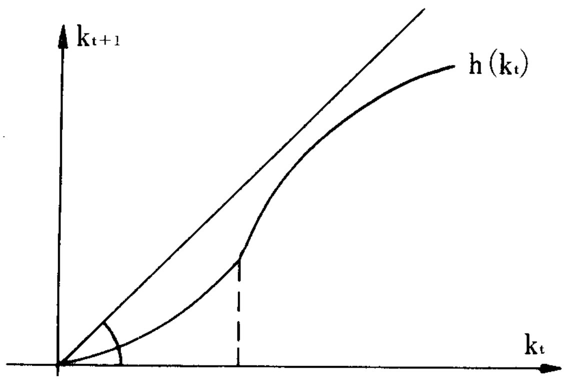
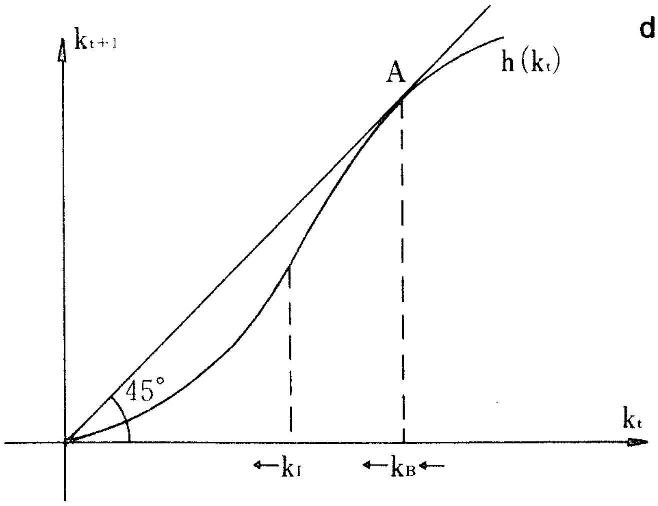
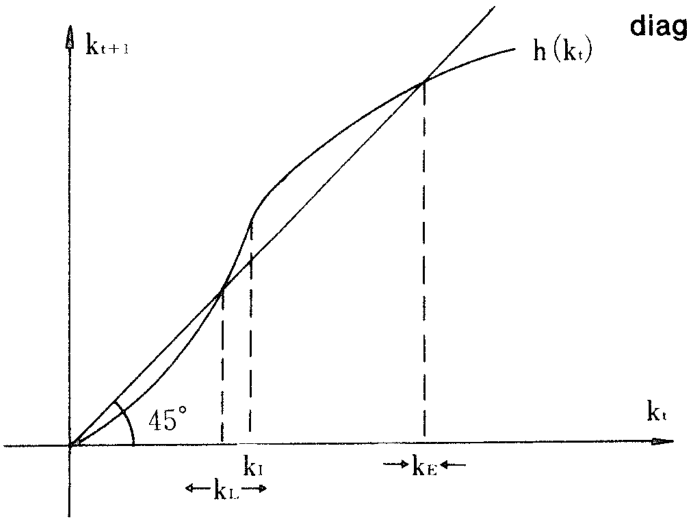


diagram 2 － 2 ②

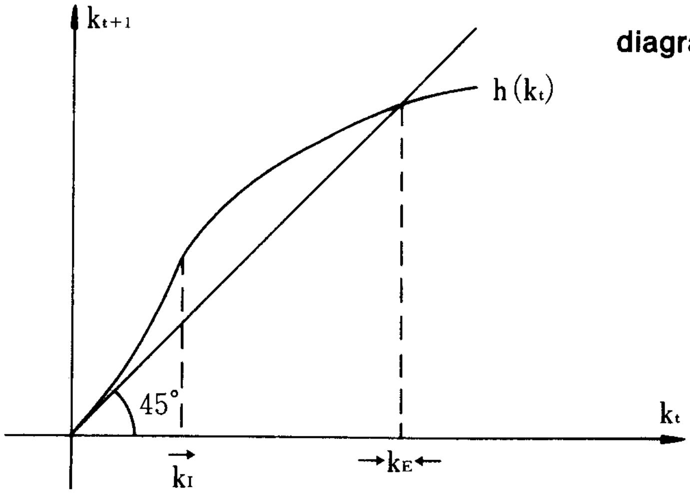


When the productivity of an economy goes up somewhat to ② of the diagram 2-2,there appears a positive steady state k_B. Any capital accumulation path starting from above k_B will converge monotonously to k_B,but any capital accumulation path starting from below k_B will converge monotonously to the origin. This case can be interpreted as follows. If the initial capital stock is increased above the level k_B for some reason such as a capital import,a sustainable stady state k_B is possible to reach.

diagram 2 － 2 ③

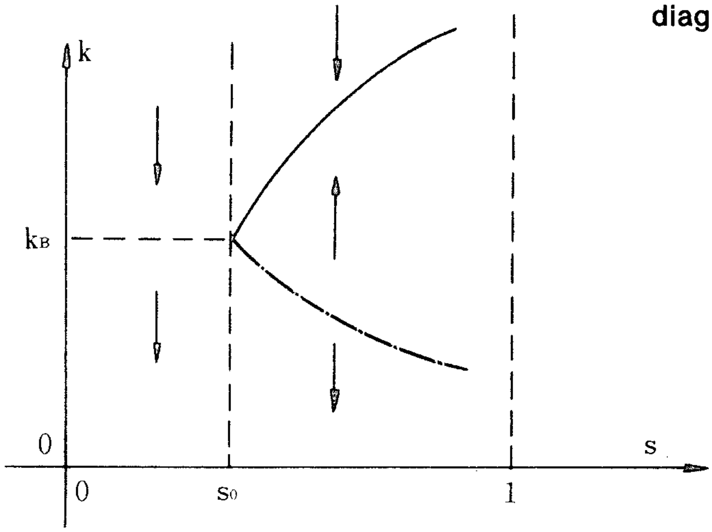


When the productivity of an economy goes up further to ③ of the diagram 2-2, there appear an unstable steady state k_L and a stable one k_E. In this case,any capital accumulation path starting from above k_L will converge to positive seady state k_E.



When the productivity of an economy goes up even further to ④ of the diagram 2-2,an unstable steady state disappears and there exists only one steady state. This phase diagram is qualitatively the same as that of the disgram 1-1.

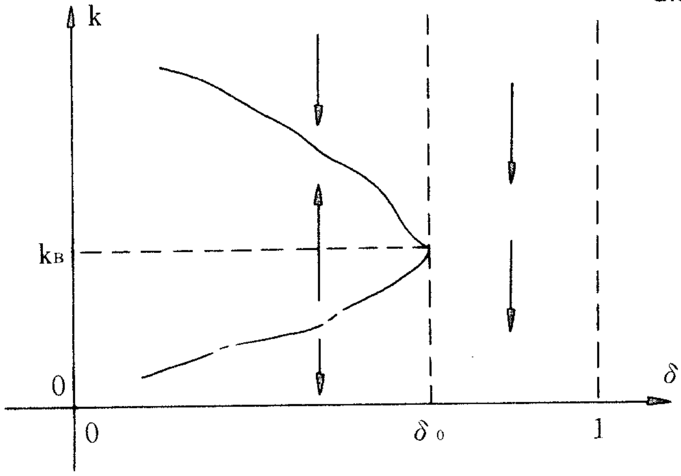
From the explanation thus far,we can see that the point A in ② of the diagram 2-2 is a turnig point. That is to say,the qualitative properties of the phase diagram is qite different between ① and ③ in this diagram. It is said that at this point A,a bifurcation occurs and the dynamical system represented by (2.3) is structually unstable. In other word,at this point, quite a small change in the parameters such as an increase of s or a decrease of δ leads the economy to the phase diagram ③,and a decrease of s or an increase of δ leads the economy to the phase diagram ①. The relationship of the changing parameter and the corresponding steady state (equilibrium position) is called a bifurcation diagram and shown in the diagram 2-3. This diagram 2-3① shows the relation between parameter s and steady state k of the capital accumulation equation that satisfies diagram 2 — 3 ①



(2.5) $k = (1-\delta)k + sg(k)$.

In this graph,the real line express the relation between stable steady state k and parameter s,the chain line represent the relation between unstable steady state k and parameter s. When the propensity to save is below s_0,the stable equilibrium capital stock is zero level. When the propensity to save is at s_0,there appears suddenly a positive steady state k_B and this equilibrium has one-sided stability. When the propensity to save gets large over s_0,there appears a branch of stable eqilibrium and unstable equiibrium. Along the same line of argument mentioned,we can see the reltionship between steady state k and parameter δ. The bifurcation graph in this cae is shown in ② of the diagram 2-3.

diagram 2 — 3 ②



Summing up arguments above,in order to escape from the vicious cycle of poverty, we need some kinds of take-off to some extent such as a rise of productivity or an increase of the propensity to save or a decrease of the depreciation rate in capital stock or some combination of these effects.

《SECTION 3》

The capital accumulation paths when the production function has a part of increasing returns;an example economy when the production function is S shape.

In this section,by specifying the production function in the section 2,we examine the capital accumulation paths and the bifurcations more concretely when the production function is S shape. That is to say,we specify the production function $g(k)$ of the preceding section as

(3.1) $g(k)=k^2, (0 \leqq k \leqq 1);\ g(k)=1+(k-1)^{1/2}, (k \geqq 1).$

It is easy to see that this production function satisfies the conditions below which are similar to (2.1).

(3.2) $g(0)=0,\ g'(k)>0,\ \lim_{k\to 0} g'(k)=0,\ \lim_{k\to\infty} g'(k)=0,$

$g''(k)>0, (0<k<1);\ g''(k)<0, (k>1).$

We can easily see that the basic accumulation function $h(k)$ defined by (2.2) becomes in this case

(3.3) $k_{t+1}=h(k_t) \equiv (1-\delta)k_t+sg(k_t)=(1-\delta)k_t+s(k_t)^2, (0 \leqq k \leqq 1),$

$k_{t+1}=h(k_t) \equiv (1-\delta)k_t+sg(k_t)=(1-\delta)k_t+s+s(k_t-1)^{1/2}, (k \geqq 1).$

We can easily ascertain that the accumulation function h(k) satisfies the properties below

(3.4) $h(0)=0,\ h(1)=1-\delta+s,\ h(2)=2(1-\delta+s),$

$\lim_{k\to+0}h'(k)=1-\delta,\ \lim_{k\to\infty}h'(k)=1-\delta,$

$h''(k)>0,(0<k<1);\ h''(k)<0,(k>1).$

The steady state in the range of $0\leqq k\leqq 1$ that satisfies the equation $k=h(k)$ is the solution of

(3.5) $k=(1-\delta)k+s(k)^2,(0\leqq k\leqq 1).$

Therfore we have $k=0$ or $k=\delta/s\leqq 1$. And the steady state in the range of $k\geqq 1$ is the solution of

(3.6) $k=(1-\delta)k+s+s(k-1)^{1/2},(k\geqq 1).$

Solving this eqution,we get

(3.7) $k=\{(1+2\alpha)\pm(1+4\alpha-4\alpha^2)^{1/2}\}/2\alpha^2,\quad \alpha\equiv\delta/s.$

Depending on the values of the prameter α,the number of intersection between the 45 degree line and the h(k) function is different. Doing the calculations, we get

(3.8) 2 intersection points $(0\leqq\alpha<(1+2^{1/2})/2)$,

tangential intersection $(\alpha=(1+2^{1/2})/2)$,

zero intersection point $(\alpha>(1+2^{1/2})/2)$.

And the value of k at which a bifurcation occurs becomes

(3.9) $k=4-2(2^{1/2}),\ (\alpha=(1+2^{1/2})/2)$

From the arguments so far,the phase diagram of capital accumulation can be shown as the diagram 3-1 depending on the values of parameters. The case ④ of this diagram represents a bifurcation. In the space of δ and s of the diagram 3−2,we can show the situation corresponding to the ranges of ①~⑤ of the diagram 3-1. For an example,on the straight line ④ of the diagram 3—2,the increase of s that surpass the increase of δ will lead the economy to the range of ③ in which a positive equilibrium is reached.

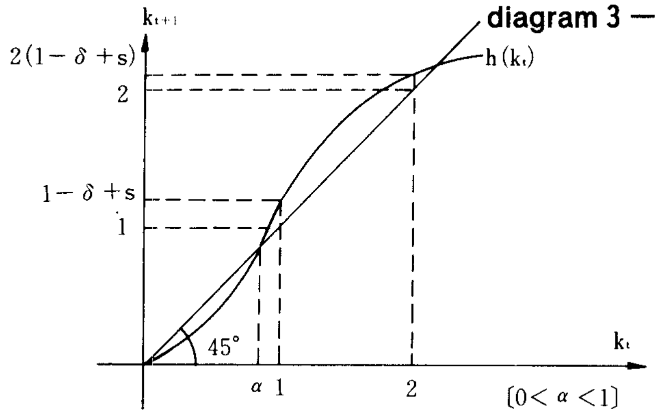


diagram 3 — 1 ①

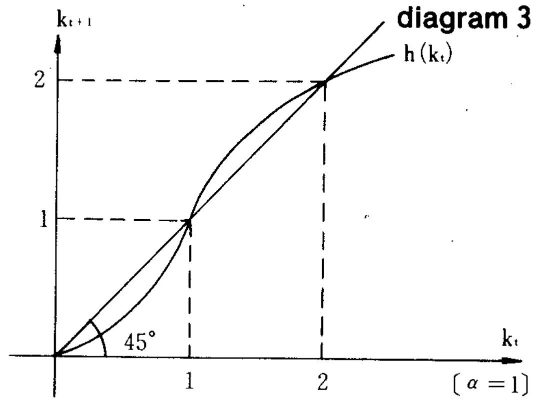


diagram 3 — 1 ②

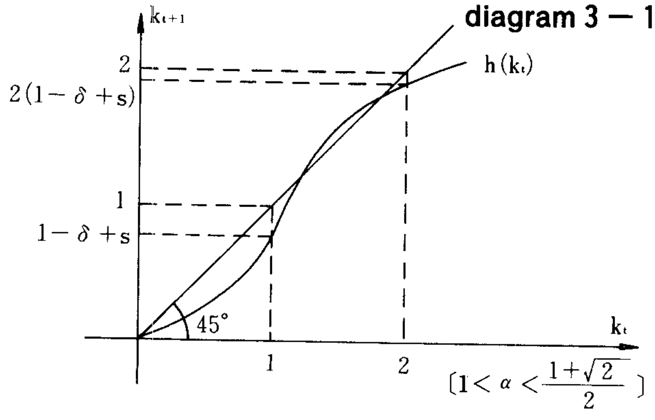


diagram 3 — 1 ③

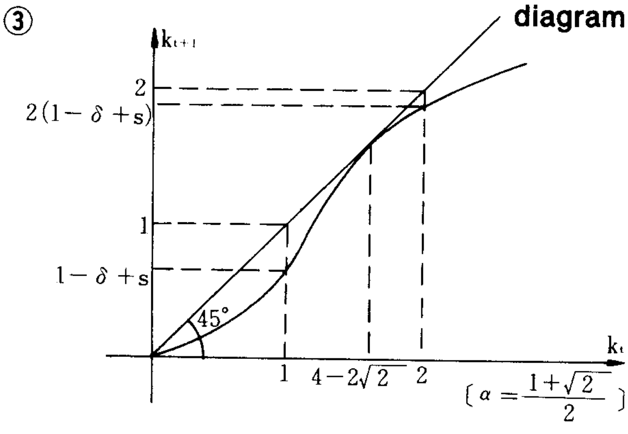


diagram 3 — 1 ④

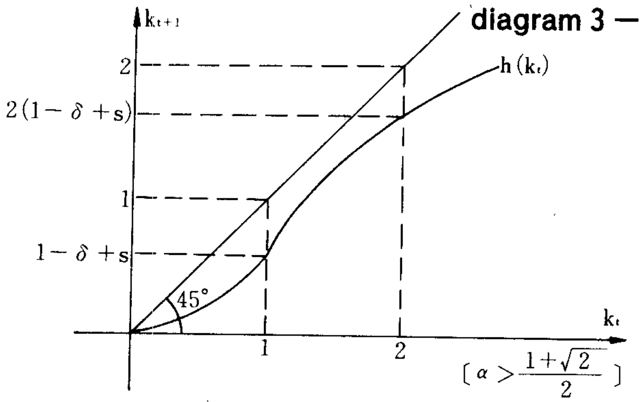


diagram 3 — 1 ⑤

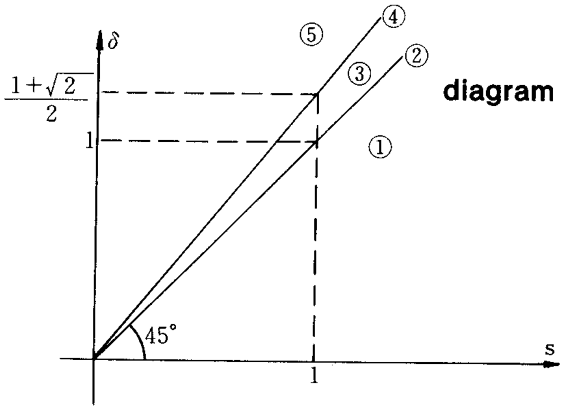


diagram 3 — 2

《SECTION 4》

The capital accumulation paths when the production function has a part of increasing returns;the production function is M shape.

In the model of the section 2,the economy which is in the trap of the vicious cycle of poverty converged to the zero level of capital stock, namely absolute poverty. But more realistic model requires that the levl of poverty is some positive level. We change the shape of the production function in the section 2 into M shape in this section. This type of the production function implies that we have an increasing returns in the middle range of capital stock level. This asumption is based on the idea that some sufficiently large level of capital is usually necessary in order to raise a productivity. That is to say,the increasing returns needs generally considerably large capital stock.

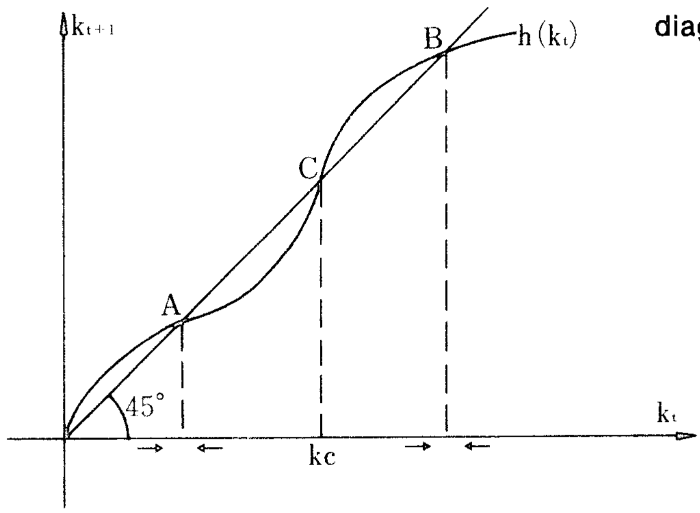
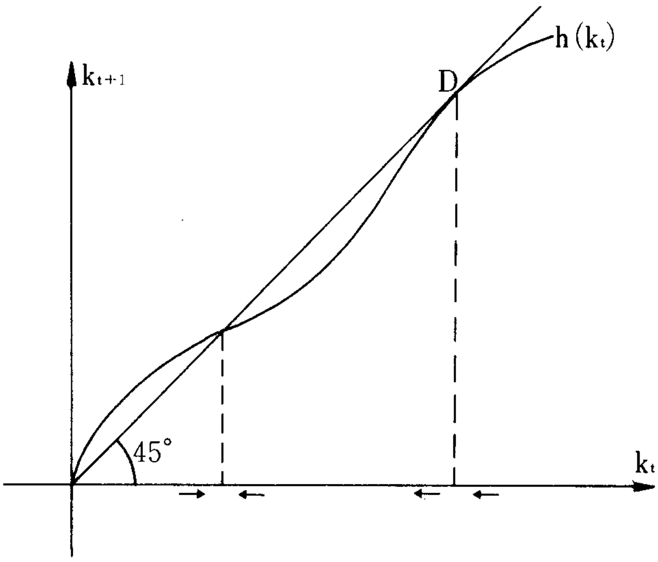


diagram 4 — 1 ①

Under this shape of the production function,the rest of assumptions being the same as that of the section 2,the phase diagram of the capital accumulation paths can be shown such as in the diagram 4-1. The graph ① of this diagram shows that there exists a critical capital stock level kc,the paths starting from below kc converges to the lower stady state A and the paths starting from above kc converges to the higher stady state B. When the productivity of the economy goes down to the graph ③ through the graph ②,the critical level kc vanishes. Any capital accumulation path converges to the lower stady state E. This fact implies that no matter how large the developping coutries import capital from abroad,it is not possible to take on a growth path without making its productivity higher to some extent.

diagram 4 — 1 ②



According to the rise of productivity,the phase graphs of the production functions follow gradually from ①,②,③,④ to ⑤ in the diagram 4-2. At first,it is easy to see that any capital accumulation path starting from between zero and A approaches to steady state A,and with the rise of the prodctivity from ① to ②,③,④ this path gets closer to the stady states B,C,D. But when the prodctivity goes up further from ④ to ⑤,the capital accumulation path we are chasing suddenly jumps and converges to steady state I without ever approaching to the steady states E,F,G,H.

diagram 4 — 1 ③

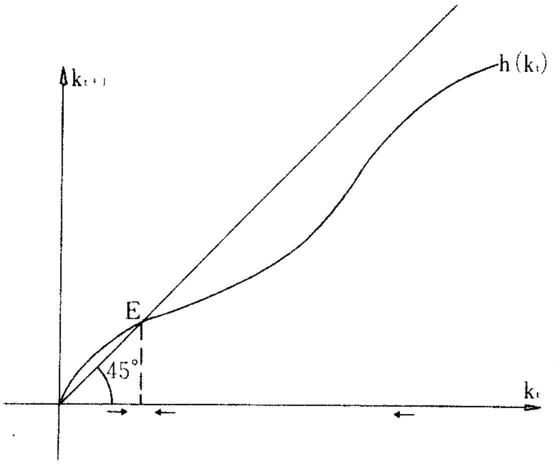
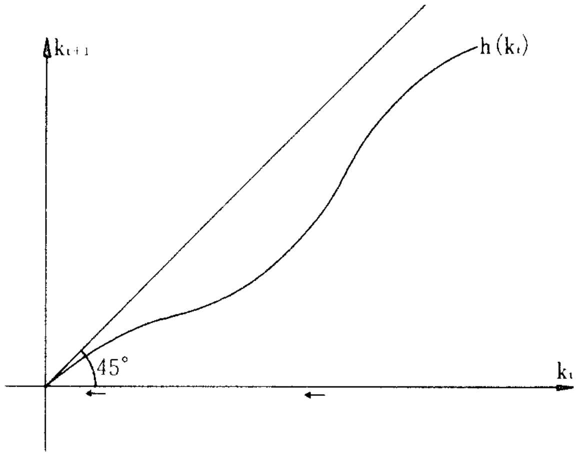


diagram 4 — 1 ④



And the revese arguments also might hold as follows. According to the decline of productivity,the phase graphs of the production functions goes down gradually from ⑤ to ④,③,②,① in the diagram 4-2. At first,it is easy to see that any capital accumulation path starting from above I approaches to steady state I,and with the decline of the prodctivity from ④to ③,② this path gets closer to the stady states H,G,F. But when the prodctivity further goes down from ② to ①,the capital accumulation path we are chasing suddenly tumbles down and converges to steady state A without ever approaching to the steady states E,D,C,B.

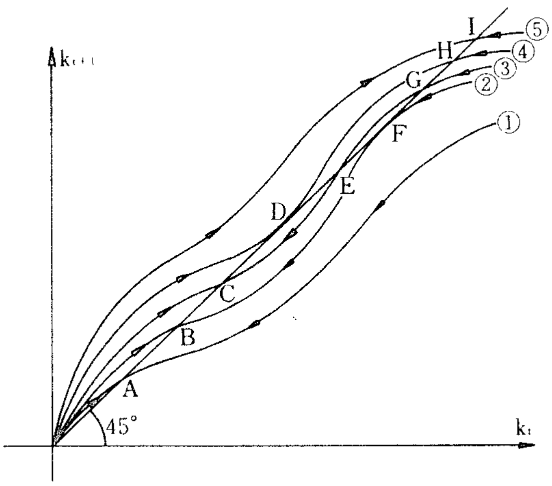


diagram 4 — 2

These arguments mentioned above imply that there might happen a sudden change of a capital accumulation path such as a sharp rise and a dive even if the productivity of the economy changes gradually.

《SECTION 5》

The accumulation paths in the overlapping generations model.

The arguments so far assumed the constant propensity to save and ignored the optimal behavior of household sector in order to make the analysis simple. In this section,we consider the accumulation paths in the overlapping generations model which involves the utility maximization behaivior of economic agents. The framework of the model is described as follows. There are two (young and old) generations at every period who live two periods and their overlapping period is 1. During young period they supply their labor inelastically to firms (old generations) and earn wages. Young generations consume goods and saves a part of their wages. These savings are used as capital stock in next period and earns rental (interest) rate. During old period,old generations consume savings and their interest,and they leave no bequest. To make analysis simple in this model,only one goods is produced and consumed. The rate of growth of labor is n,that is to say $L_{t+1}=(1+n)L_t$. The utility function of a representative household of generations t is $u(c^1_t,c^2_{t+1})$,where c^1_t and c^2_{t+1} are respectivly the consumption of young at t and the consumption of old at t +1. The $u(\cdot)$ is assumed to be a strictly qasiconcave function. The production function F (K,L) is a continuosly differentiable,linearly homogenous neoclassical function. Expressing this function per labor unit,f(k) is assumed to satisfy

(5.1) $f(0)=0,\ f'(k)>0,\ f''(k)<0\ ,\lim_{k\to 0}f'(k)=\infty,\ \lim_{k\to\infty}f'(k)=0.$

Under the competitive factor markets,the factors recieve their income corespponding to their marginal productivity,namely

(5.2) $r=f'(k)\equiv r(k),\ w=f(k)-kf'(k)\equiv w(k),$

where r and w are respectively the rental rate of capital and the rate of wage in terms of goods. The functions r(k) and w(k) thus defined satisfy from (5.2)

(5.3) $r'(k)<0,\ w'(k)>0.$

The budget equation of the member of generation t is

(5.4) $s^t=w_t-c^1_t,\ c^2_{t+1}=(1+r_{t+1}-\delta)s_t,$

where s_t is the savings of the member of generation t when young. Under (5.4) a household tries to maximize utility,that is to say,the maximization of $u(w_t-s_t,(1+r_{t+1}-\delta)s_t)$ in s_t. From this optimization, the saving function s(・) is derived.

(5.5) $s_t=s(1+r_{t+1}-\delta,w_t)=s(1+r(k_{t+1})-\delta,w(k_t)).$

The savings in the economy at period t becomes L_ts_t,and these resources are used as the caital stock in next period t+1,consequently we have the relation of

(5.6) $K_{t+1}=L_ts_t.$

Arranging this equation in terms of per labor unit,we get

(5.7) $k_{t+1}=s_t/(1+n)=\{s(1+r(k_{t+1})-\delta,w(k_t))\}/(1+n).$

This is the basic capital accumulation equation in the overlapping generations model.

In order to solve (5.7) on k_{t+1} as a function of k_t,we need the implicit function theorem and the sufficient condition for this is

(5.8) $1+n-s_r r' \neq 0$,

where s_r is the partial derivative of $s(\cdot)$ on the rate of yeild $(1+r-\delta)$. If we assume $s_r \geqq 0$,it is clear that (5.8) is satisfied.

And we also get from the implicit function theorem

(5.9) $dk_{t+1}/dk_t = s_w w'/(1+n-s_r r')>0$,

where s_w is the partial derivative of $s(\cdot)$ on the wage. We assume that the next period goods are normal goods,and this implies (5.9).

From (5.7) and (5.8),the capital accumulation function $h(\cdot)$ is derived as below

(5.10) $k_{t+1}=h(k_t)$,

and k_{t+1} is an increasing function of k_t from (5.9). This is the basic capital accumulation function in the overlapping generations model.

《SECTION 6》

The capital accumulation paths in the overlapping generations model;an example economy.

In this section,we specify the production function and the utility function for the model in the section 5,and examine the properties of the capital accumulation paths. The rest of analitical framework is the same as that of the section 5.

Concretely speaking

(6.1) $u(c^1_t, c^2_{t+1}) = (1-\beta)\log(c^1_t) + \beta\log(c^2_{t+1}), (\beta>0)$, (6.2)　$f(k_t) = a\log(1+k_t), (a>0)$

are the specifications[5] of these functions in this section,where β and a are parameters.

Under the budget eqation of (5.4),we maximize (6.1) and get the saving function

(6.3)　$s_t = \beta w_t$,

where β in this case evidently represents the propensity to save. The wage function is also derived from (5.2) and (6.2),and we get

(6.4)　$w(k) = f(k) - kf'(k) = a\log(1+k) - ak/(1+k)$.

It is easy to see that this wage function satisfies

(6.5)　$w'(k) = -kf''(k) = ak/(1+k)^2,\ w''(k) = -f''(k) - kf'''(k) = a(1-k)/(1+k)^3$.

From the arguments in section 5,we can see that the basic capital accumulation function h(k) becomes in this case

(6.6)　$k_{t+1} = h(k_t) = \beta w(k_t)/(1+n) = \beta a\{\log(1+k_t) - k_t/(1+k_t)\}/(1+n)$.

The first and second derivatives of h(k) are easily calculated as shown below

(6.7)　$h'(k_t) = dk_{t+1}/dk_t = \beta w'(k_t)/(1+n) = \beta ak_t/\{(1+n)(1+k_t)^2\}$,

(6.8)　$h''(k_t) = \beta w''(k_t)/(1+n) = \beta a(1-k_t)/\{(1+n)(1+k_t)^3\}$.

From these equations,we can easily see

(6.9)　$\lim_{k\to 0} h'(k) = 0,\ \lim_{k\to\infty} h'(k) = 0,$

$h''(k) > 0, (0<k<1);\ h''(k) = 0, (k=1);\ h''(k) < 0, (k>1)$.

We specified the production function as (6.2),and this is concave function. It is to be noted that even if the production function has no part of increasing returns,the capital accumulation function h(k) has a part of 凹 and 凸 such as shown in (6.9). From (6.6) we can also caluculate

(6.10) $h(1) = a\beta(\log 2 - 1/2)/(1+n)$.

And the value of k at which a bifurcation occurs can be calculated from the equations below

(6.11) $k = a\beta\{\log(1+k) - k/(1+k)\}/(1+n)$

(6.12) $h'(k) = a\beta k/\{(1+n)(1+k)^2\}$

The phase diagram in this sction can be shown as the diagram 6-1. The case ② in this diagram is the situation of bifurcation. The slight changes of parameters such as an increase of β or an increase of a or a decrease of n will lead the economy to the case ③ in the diagram. And the slight changes of parameters such as a drcrease of β or a decrease of a or an increase of n will lead the economy to the case ① in the diagram. From the arguments mentioned above,we can see that in order to have a positive stable equilibrium it is necessary to make the productivity or the propensity of saving rise or to make the labor growth decline.

diagram 6 — 1 ①

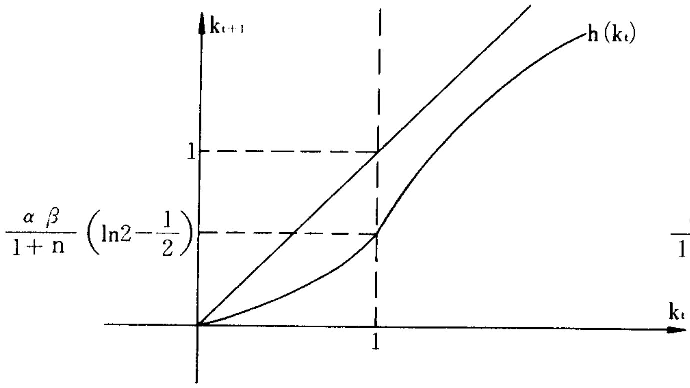


diagram 6 — 1 ②

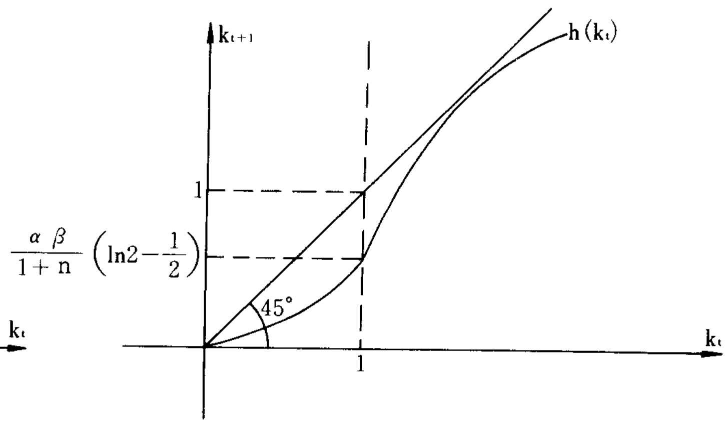
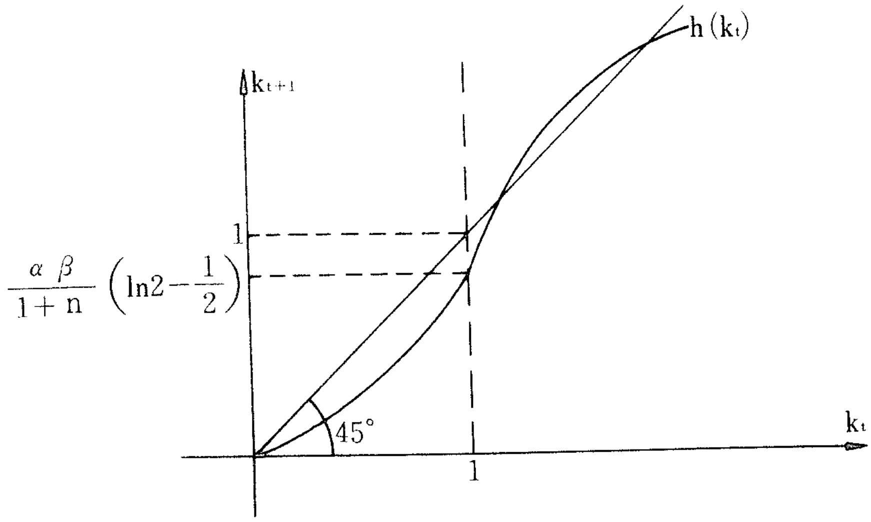


diagram 6 − 1 ③



* This article is basically the Englilsh translation of my paper which is going to be published in Western Japanese Journal of Theoretical Economics in this spring. There is only a few additions and revisions to my early paper.

《NOTE》

(1) We can refer to the paper of Baumol,Benhabib (1989) as a readable application of chaos theory to economics.

(2) As a survey on the bifurcation theory,we can refer to Nishimura, Yano (1993) and Azariadis (1993).

(3) We might roughly say that among European journal papers such as H.W.Lorenz (1993) the dynamic macroeconomics is based on Keynesian economics and among American journal papers it is based on neoclassical growth theory.

(4) It is a pity that there are many type errors in both Nishimura, Yano (1993) and Azariadis (1993).

(5) The example of this production function is treated in p.94 of Azariadis (1993).

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